

String Corrected Spacetimes and $SU(N)$ -Structure Manifolds

Katrin Becker, Melanie Becker, and Daniel Robbins

*George P. and Cynthia W. Mitchell Institute for
Fundamental Physics and Astronomy, Texas A&M University,
College Station, TX 77843-4242, USA*

Submitted to Nuclear Physics B Special Issue: 60 Years of Calabi Conjecture

Abstract

Using an effective field theory approach and the language of $SU(N)$ -structures, we study higher derivative corrections to the supersymmetry constraints for compactifications of string or M-theory to Minkowski space. Our analysis is done entirely in the target space and is thus very general, and does not rely on theory-dependent details such as the amount of worldsheet supersymmetry. For manifolds of real dimension $n < 4$ we show that internal geometry remains flat and uncorrected. For $n = 4, 6$, Kähler manifolds with $SU(N)$ -holonomy can become corrected to $SU(N)$ -structure, while preserving supersymmetry, once corrections are included.

Contents

1	Introduction	2
2	Two dimensions	6
3	Three dimensions	9
4	Four dimensions	10
4.1	Spinors and bispinors	11
4.2	Metric from SU(2)-structure	13
4.3	Decomposition of forms	13
4.4	Gravitino SUSY variation and background geometry	14
5	Six dimensions	19
5.1	Spinors and bispinors	19
5.2	Metric from SU(3)-structure	21
5.3	Decomposition of forms	22
5.4	Gravitino SUSY variation and background geometry	23
A	Useful formulae	25

1 Introduction

Ever since the discovery of the relevance of Calabi-Yau manifolds in the context of string theory [1] physicists have wondered if corrections (either perturbative or non-perturbative) to the internal geometry M would spoil the property that Kähler manifolds with vanishing first Chern class $c_1(M) = 0$ emerge as a solution to the theory. The vanishing of the first Chern class is a topological condition that holds, in particular, when the Kähler manifold admits a Ricci-flat metric. In fact, Calabi conjectured [2] and Yau proved [3] many years ago that a compact Kähler manifold with $c_1(M) = 0$ admits a Kähler metric with SU(N)-holonomy in the same Kähler class.

Since a spinor representation of Spin($2N$) always contains a singlet when the group is reduced to SU(N), one can always find a covariantly constant spinor on SU(N)-holonomy manifolds

$$\nabla_a \eta = 0. \tag{1.1}$$

Here and in the following a, b, \dots denote the coordinates on M . This also implies that such geometries are Ricci-flat. To see this, note that the integrability condition for (1.1) is

$$[\nabla_a, \nabla_b] \eta = 0. \tag{1.2}$$

This leads to the equation

$$R_{abcd}\Gamma^{cd}\eta = 0, \quad (1.3)$$

which by contraction with Γ^b results in a vanishing Ricci tensor, $R_{ab} = 0$.

How is this discussion related to the internal geometries and how do we take string theory corrections into account? This is the underlying idea discussed in this paper.

The basic situation we have in mind is that we are starting with some effective space-time theory in D dimensions formulated in terms of a derivative expansion. We consider the effect that string theory corrections (either perturbative or nonperturbative) have on the vacuum solution. More precisely, we are looking for supersymmetric solutions which have the form of a product of $(D - n)$ -dimensional Minkowski space times an n -dimensional internal space M . To find such a solution, we must ensure that the supersymmetry variations of the fermionic fields vanish for some choice of nowhere-vanishing anticommuting supersymmetry parameter ϵ .

Since we are looking for manifolds with a Minkowski space factor, it is natural to decompose the $\text{Spin}(D - 1, 1)$ spinor ϵ into $\text{Spin}(D - n - 1, 1) \times \text{Spin}(n)$ schematically¹ as $\epsilon = \xi \otimes \eta$, where ξ is an anticommuting constant spinor on $\mathbb{R}^{D-n-1,1}$ and η is a commuting spinor on the manifold M .

The vanishing of the SUSY variation of the gravitino gives, to leading order, precisely the condition (1.1) mentioned earlier

$$0 = \delta\psi_A = \nabla_A \epsilon \iff \nabla_a \eta = 0. \quad (1.4)$$

Here the covariant derivative on the right is with respect to the metric on M and indices with capital letters are ten dimensional. Since we have turned off fields other than the metric, the remaining fermionic variations will automatically vanish, as will other terms in the gravitino variation².

If we now include higher order corrections in our effective theory, the equation we need to solve becomes

$$\nabla_a \eta = X_a[g]\eta. \quad (1.5)$$

¹Depending on exactly what dimensions and which spinor representations are involved, we may need to include a couple of terms in this decomposition. For instance, if all the dimensions are even and ϵ is positive chirality, then we would have $\epsilon_+ = (\xi_+ \otimes \eta_+) \oplus (\xi_- \otimes \eta_-)$, where subscripts indicate chiralities. But these subtleties will not concern us in setting up this general story.

²This is the simplest situation, but our techniques can be applied more broadly. For instance, in string theory, fluxes are quantized in string units. Thus if we turn on a fixed number of flux quanta, but can access a large volume limit (i.e. the volume of the internal space is large in string units), then the magnitude of the flux density scales as a positive power of α' and can be moved to subleading order in the perturbation series. This is sometimes called the *dilute flux approximation*. Note, however, that we should still make sure that other SUSY variations vanish, such as the dilatino variation. We won't pursue this line of attack in the current paper.

Here $X_a[g]$ is some matrix³ constructed covariantly from the metric g_{ab} on M . This matrix encodes the string theory corrections, whose form depends on the dimension of M and the precise theory being considered, as we see later on. In summary, we want to find a metric g_{ab} on M , and a nowhere-vanishing spinor η , such that (1.5) is satisfied. From our previous discussions we see that the corrected (physical) metric may no longer necessarily have a reduced holonomy group, nor is it necessarily Ricci-flat.

Here we only need η to be nowhere-vanishing, but of course, we can always use this to construct a unit normalized spinor $\eta' = \eta / \sqrt{\eta^\dagger \eta}$ satisfying $\eta'^\dagger \eta' = 1$. Then η' obeys an equation of similar form to (1.5), but with X_a shifted by a term proportional to $(\eta'^\dagger X_a \eta')$ times the identity matrix. In other words, η' obeys

$$\nabla_a \eta' = X'_a[g, \eta'] \eta'. \quad (1.6)$$

Henceforth we will assume that we have normalized our spinor in this way, and we will drop the primes.

Our goal is to solve (1.6), and learn about the properties of the internal geometry of M along the way. Note that this equation is very nonlinear in the metric, so solving it can be quite non-trivial. In fact, no explicit analytic expressions for Calabi-Yau metrics are known yet (see the discussion of [4] in this volume). One reasonable approach is to proceed order by order in an expansion. At each step the lower order solution will act as source terms to solve for the next order.

A useful alternative to solving directly for g_{ab} and η involves the construction of bilinears. In each dimension we can construct various differential forms on M by sandwiching antisymmetrized products of gamma matrices between η^T and η or their complex conjugates. In fact, we can typically find an invertible mapping between $\{g_{ab}, \eta\}$ and some subset of bilinears, possibly with constraints.

That is, given g and η we can of course construct our bilinears, but conversely given some subset of the bilinears, possibly obeying certain constraints, we can construct a metric and a normalized spinor (if the bilinears were known explicitly). We will see this in more detail in each dimension below. Using this map, we can convert the equation (1.6) into an equivalent set of equations on these bilinears which can often be much easier to work with. Again the details are very dimension dependent and are explored below.

³In fact, $X_a[g]$ could even in principle involve differential operators acting on η . For instance we could imagine some higher order correction in string theory leading to a term like $(\alpha')^m R^m \nabla_a \eta$ appearing on the right hand side of the corrected gravitino variation (where R^m is some scalar constructed from m Riemann tensors). This would make the discussion a little more involved but would not qualitatively change the situation, nor would it invalidate the approach being presented.

Note that the existence of a normalized spinor on the manifold M reduces the structure group from $GL(n)$ to some smaller group G , i.e. we can construct an atlas of patches \mathcal{U}_i on M where the transition functions mapping between $T\mathcal{U}_i$ and $T\mathcal{U}_j$ are elements of $G \subset GL(n)$.

Since η and g_{ab} are equivalent to some collection of forms - the bilinears discussed above - then this set of forms also accomplishes the same reduction of the structure group down to G . A specification of such a set of forms is called a G -structure. Our task will be to convert (1.6) into equations for a G -structure. More concretely, we show that in M-theory/string theory the internal manifold has $SU(N)$ -structure once higher order corrections to the space-time effective action are taken into account. We explicitly calculate the $SU(N)$ -structure for manifolds of real dimension $n = 2, 3, 4, 6$. As already mentioned, Calabi-Yau manifolds are Ricci-flat and Kähler (at least to leading order). This means there is a real two form J , the Kähler form, which is closed. Some corrections may allow for a non-vanishing dJ , so the internal geometry is no longer Kähler and yet preserves supersymmetry. We discuss this possibility in detail for $n = 4$. In addition there is a holomorphic complex form Ω which, as opposed to the leading order Calabi-Yau case, may no longer be closed. Hence the term $SU(N)$ -structure manifolds, as opposed to $SU(N)$ -holonomy manifolds.

Perturbative corrections to Calabi-Yau compactifications of string theory were considered in the literature many years ago. See [5], [6], and [7] for a partial list of references considering the effective action approach. A detailed analysis of the effect that these corrections have on a Calabi-Yau background was performed in the sigma model by [8] (see also [9] and [10]). The novelty of our work, is that it makes contact to the more recent literature about string theory compactification on $SU(N)$ -structure manifolds (see [11], [12], [13], [14], [15], [16] and [17] for a partial list of references, and see especially [18] for another work which studies string corrections in the language of G -structures, though in a somewhat different context). The approach taken here is closely related to the effective field theory approach for M-theory and type II theory compactifications on G_2 -structure manifolds and $Spin(7)$ -structure manifolds performed recently in [19]. In this case, a conventional non-linear sigma model approach is still not available [8].

Finally, we note that the one dimensional ($n = 1$) case is too simple because spinors have only one real component. Thus a unit normalized spinor just has a constant component whose absolute value is equal to one. There are no corrections allowed and the spinor is covariantly constant. The spin connection vanishes automatically in one dimension and the space (we assume compactness) is always S^1 . So the first case which isn't trivial is two dimensions. This is the case we work out next.

2 Two dimensions

Our goal in this section is to show that T^2 is the only compact internal two-dimensional geometry allowed by supersymmetry. Even though such a proof seems fairly straightforward, the explicit construction of the G-structure is very instructive. In two Euclidean dimensions we can pick gamma matrices Γ_a which are real and symmetric, and we can define the chirality operator

$$\Gamma = \frac{i}{2!} \epsilon^{ab} \Gamma_{ab}, \quad (2.1)$$

which is imaginary, antisymmetric and squares to one. Here ϵ^{ab} is the antisymmetric tensor with entries $\pm 1/\sqrt{g}$.

Suppose we have a nowhere vanishing real spinor η . This reduces the structure group from $\text{Spin}(2)$ to $\{1\}$ (the trivial group). We are free to normalize the spinor so that $\eta^T \eta = 1$ pointwise. Then, again pointwise, one choice of basis for the two-dimensional space of real spinors is $\{\eta, i\Gamma\eta\}$ which obeys the completeness relation

$$1 = \eta\eta^T + \Gamma\eta\eta^T\Gamma. \quad (2.2)$$

We can define one-forms in terms of these basis elements

$$u_a = \eta^T \Gamma_a \eta, \quad v_a = i\eta^T \Gamma_a \Gamma \eta, \quad (2.3)$$

which are related by a duality transformation

$$v = - * u. \quad (2.4)$$

Using the completeness relation, one can show that u_a and v_a are zweibeine.

$$u_a u_b + v_a v_b = g_{ab}. \quad (2.5)$$

Here and in the following sections g_{ab} denotes the metric on M . One can show (by contracting (2.5) in various ways), that the zweibeine satisfy

$$u^a u_a = v^a v_a = 1, \quad u^a v_a = 0. \quad (2.6)$$

We can either proceed by working with the metric and the spinor, or equivalently use an approach in terms of the zweibeine. In fact, specifying a pair of globally defined real one-forms u_a and v_a such that $u \wedge v$ is nowhere vanishing is (almost⁴) equivalent to specifying a metric and a unit normalized spinor η . In terms of counting components, we have either three components

⁴There are two caveats here if we want to be completely precise. First of all as explained below, there's also the choice of orientation to be made. Secondly, one can see that sending $\eta \rightarrow -\eta$ leaves u and v unchanged, so there's an additional \mathbb{Z}_2 covering issue. Neither of these subtleties will cause us problems, but it is good to keep them in mind.

from g_{ab} and one from η , or we have two each from u_a and v_a . In either case the total number (four) parameterizes the space of G -structures ($G = \{1\}$ here), which indeed should have dimension

$$\dim(\mathrm{GL}(2)) - \dim(\{1\}) = 4 - 0 = 4. \quad (2.7)$$

The map between this data in one direction is given by (2.3), because the spinor and the gamma matrices (alias the metric) determine the zweibeine. For the other direction we can define a metric by (2.5). Explicitly, the inverse metric is given by

$$g^{-1} = \frac{1}{(u_1 v_2 - u_2 v_1)^2} \begin{pmatrix} (u_2)^2 + (v_2)^2 & -u_1 u_2 - v_1 v_2 \\ -u_1 u_2 - v_1 v_2 & (u_1)^2 + (v_1)^2 \end{pmatrix}. \quad (2.8)$$

From this expression one can directly show the various contraction identities (2.6). Further, an appropriate choice of orientation (we need to pick the epsilon tensor to satisfy $\epsilon_{ab} = u_a v_b - v_a u_b$) will ensure that (2.4) holds as well.

What about the spinor η ? If we pick a choice of flat gamma matrices, say for instance

$$\Gamma_{\underline{u}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_{\underline{v}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.9)$$

(where an underline refers to a flat index), then the curved gamma matrices are expressed in terms of the zweibeine

$$\Gamma_a = u_a \Gamma_{\underline{u}} + v_a \Gamma_{\underline{v}} = \begin{pmatrix} u_a & v_a \\ v_a & -u_a \end{pmatrix}. \quad (2.10)$$

The explicit form of the chirality operator is

$$\Gamma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (2.11)$$

Now we need to pick a spinor η that satisfies (2.3). It is easy to check that the solution is $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (or $\eta = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$). This seems surprising at first, since there should have been one free component of η to match our counting. Looking more carefully, we note that there is one component of u_a and v_a which does not enter g_{ab} , and which corresponds to the freedom to make local $\mathrm{SO}(2)$ rotations. When we make such a rotation we are essentially changing the choice of flat gamma matrices (2.9), and this choice gets translated immediately into the missing component of η . Explicitly, if we take

$$\Gamma'_{\underline{u}} = \cos \phi \Gamma_{\underline{u}} + \sin \phi \Gamma_{\underline{v}}, \quad \Gamma'_{\underline{v}} = -\sin \phi \Gamma_{\underline{u}} + \cos \phi \Gamma_{\underline{v}}, \quad (2.12)$$

with ϕ being some function on M , then the corresponding η' would be given by

$$\eta' = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}. \quad (2.13)$$

This shows the equivalence between the variables $\{g_{ab}, \eta\}$ and $\{u_a, v_a\}$.

Working in terms of the metric/spinor pair, we can make the following ansatz for the corrected SUSY variation of the gravitino

$$\nabla_a \eta = A_a \eta + i B_a \Gamma \eta. \quad (2.14)$$

Here we have used that any spinor can be expanded into our basis (2.3), with coefficients A_a, B_a that serve as source terms. It is easy to see though, that $A_a = 0$ for a spinor with unit norm (since $0 = \nabla_a(\eta^T \eta) = 2A_a$) and that $A_a = B_a = 0$ for the vacuum. Let's understand how to solve this equation for the metric/spinor pair and then we'll equivalently think about the question in terms of the one-forms u and v . The integrability condition obtained by hitting (2.14) by another nabla and antisymmetrizing takes the form

$$\nabla_{[a} B_{b]} = -\frac{1}{8} \epsilon^{cd} R_{abcd} = -\frac{R}{8} \epsilon_{ab}, \quad (2.15)$$

where we used the fact that in two dimensions

$$R_{abcd} = \frac{R}{2} (g_{ac} g_{bd} - g_{ad} g_{bc}). \quad (2.16)$$

Since B_a encodes the (perturbative and possibly non-perturbative corrections), it has an expansion in derivatives, vanishing at lowest order. Then at lowest order (2.15) implies that $R_{abcd} = 0$, so the space is (assuming also compactness and orientability) simply T^2 with a flat metric. We are free to choose coordinates in which the metric is simply constant. In these coordinates the spin connection vanishes and so we see that η must also be constant. At the next order (and every higher order), B_a will be some local covariant construction in terms of curvatures and derivatives, which vanishes when evaluated on T^2 . Thus $B_a = 0$ to all orders in perturbation theory (and even non-perturbatively) and the solution remains a flat torus. For the same reason the spinor is constant to all orders.

How do we see this if we translate the problem to $\{u, v\}$? By taking derivatives of (2.3) we obtain

$$\nabla_a u_b = 2B_a v_b, \quad \nabla_a v_b = -2B_a u_b. \quad (2.17)$$

As it is these equations still depend on the metric through the covariant derivatives. But by antisymmetrizing, we obtain

$$du = 2B \wedge v, \quad dv = -2B \wedge u. \quad (2.18)$$

If we now view u_a and v_a as independent quantities, and B_a is a functional of u_a and v_a (via the metric $g_{ab}[u, v]$ as an intermediary), then (2.18) are equations that we need to solve. In fact it will turn out that these equations are equivalent to (2.14), as will become clear by our solution.

Recalling the identification of u and v as zweibeine, (2.18) identifies B_a as the spin connection,

$$B_a = -\frac{1}{2}(\omega_a)^{\underline{u}}_{\underline{v}}. \quad (2.19)$$

Both sides of this equation should be viewed as functionals of u and v , and the equation itself is a constraint on u and v . At leading order $B_a = 0$, so we have vanishing spin connection and hence our manifold is T^2 with a flat metric. These statements hold to all orders in the perturbative expansion (and even non-perturbatively) because $B_a = 0$, when evaluated in the flat background solution. We can then choose u and v to be the zweibeine compatible with these statements. Note that given such a choice of u and v , we can always add exact one-forms and still have a solution (at least if the deformation is small enough to not spoil the condition $u \wedge v \neq 0$). This corresponds to performing a diffeomorphism, and we shall see an analog to this freedom in each dimension.

3 Three dimensions

We will be somewhat briefer in this case, because the arguments work along the same lines as in two dimensions. In this case T^3 is the only compact manifold allowed by supersymmetry, as we discuss next. In three Euclidean dimensions, the two-by-two gamma matrices Γ_a cannot be chosen to be real or have particular symmetry properties (other than being hermitian). There is a unitary charge conjugation matrix C which is antisymmetric and satisfies $\Gamma_a^T = -C^{-1}\Gamma_a C$. For example, if we take the standard Pauli matrices σ_a for Γ_a , then we can take $C = \sigma_2$. There is a relation

$$\frac{i}{3!}\epsilon^{abc}\Gamma_{abc} = 1. \quad (3.1)$$

Suppose we have a nowhere vanishing complex spinor η . This reduces the structure group from $\text{Spin}(3)$ to $\{1\}$. We can normalize $\eta^\dagger \eta = 1$, which still leaves an unfixed phase of η . A complex basis for the spinors is then given by η and $C\bar{\eta}$ (the bar denoting complex conjugation), the latter also being normalized $(C\bar{\eta})^\dagger (C\bar{\eta}) = 1$, since C is unitary. We also have orthogonality, $\eta^\dagger (C\bar{\eta}) = 0$, by the antisymmetry of C . The corresponding completeness relation satisfied by these basis elements is

$$1 = \eta\eta^\dagger + C\bar{\eta}(C\bar{\eta})^\dagger. \quad (3.2)$$

In terms of this basis there are three real one forms we can construct

$$u_a = \eta^\dagger \Gamma_a \eta, \quad (3.3)$$

and

$$v_a = \frac{1}{2} \left[(C\bar{\eta})^\dagger \Gamma_a \eta + \eta^\dagger \Gamma_a (C\bar{\eta}) \right], \quad w_a = \frac{i}{2} \left[(C\bar{\eta})^\dagger \Gamma_a \eta - \eta^\dagger \Gamma_a (C\bar{\eta}) \right]. \quad (3.4)$$

The remaining one-form is not independent, since

$$(C\bar{\eta})^\dagger \Gamma_a (C\bar{\eta}) = \eta^T C^{-1} \Gamma_a C \bar{\eta} = -\eta^T \Gamma_a^T \bar{\eta} = -u_a. \quad (3.5)$$

This triplet of one-forms are an equivalent parametrization of the data of the G -structure. Indeed, we can check that counting matches, since

$$(\# \text{ of real components of } g \text{ and } \eta) = 6 + 3 = 9, \quad (3.6)$$

$$(\# \text{ of components of } \{u, v, w\}) = 3 \times 3 = 9, \quad (3.7)$$

and the space of G -structures has dimension

$$\dim(\text{GL}(3)) - \dim(\{1\}) = 9 - 0 = 9. \quad (3.8)$$

The one-forms behave as a set of dreibeine, since we can show using (3.2) that

$$u_a u_b + v_a v_b + w_a w_b = g_{ab}. \quad (3.9)$$

Working with the metric/spinor pair is again equivalent to working with the dreibeine defined above. In the former approach, we write the susy transformation of the gravitino in terms of our basis of spinors

$$\nabla_a \eta = i A_a \eta + B_a C \bar{\eta}. \quad (3.10)$$

Since we should preserve the normalization $\eta^\dagger \eta = 1$, it follows that A_a here should be real, while B_a may be complex. In terms of the dreibeine introduced above, A_a and B_a are the nine real components of the spin-connection. Since A_a and B_a vanish to lowest order, the unique compact solution is a flat torus T^3 and a constant complex spinor. By the same arguments as in two dimensions, A_a and B_a vanish to all orders when evaluated on the background T^3 . So a flat three dimensional torus is the unique solution to all orders in perturbation theory and non-perturbatively. Similarly, the spinor is constant to all orders. We observe that to leading order the dreibeine, u , v , and w are all Killing vectors and the associated metric is flat. By the same argument that we used in two dimensions, this holds to all orders. As in two dimensions, infinitesimal diffeomorphisms are incorporated by the freedom to add arbitrary exact one-forms to u , v , and w .

4 Four dimensions

In this dimension, corrections can change the geometry in a more interesting manner. Supersymmetry allows for two compact manifolds, as we discuss next.

4.1 Spinors and bispinors

In four dimensions, there is again no basis in which the 4×4 Γ_a can be chosen to be real or pure imaginary. We define the chirality operator

$$\Gamma = \frac{1}{4!} \epsilon^{abcd} \Gamma_{abcd}, \quad (4.1)$$

which satisfies $\Gamma^2 = 1$. In fact we can choose a basis in which Γ is real and symmetric. Note also that

$$\Gamma \Gamma_{ab} = -\frac{1}{2} \epsilon_{abcd} \Gamma^{cd}. \quad (4.2)$$

The unitary charge conjugation matrix C is antisymmetric, satisfies⁵ $\Gamma_a^T = -C^{-1} \Gamma_a C$ and commutes with Γ .

Suppose we have a nowhere vanishing complex spinor η with a particular chirality, say⁶ $\Gamma \eta = -\eta$, which we can normalize so that $\eta^\dagger \eta = 1$. This reduces the structure group from $\text{Spin}(4)$ to $\text{SU}(2)$. Then, analogous to three dimensions, η and $C\bar{\eta}$ build a complex basis for the positive-chirality spinors, and we have a corresponding completeness relation

$$\frac{1 - \Gamma}{2} = \eta \eta^\dagger + C\bar{\eta} (C\bar{\eta})^\dagger. \quad (4.3)$$

There are no real one-forms that we can construct in this case, but there are three real two-forms, or one real and one complex two-form,

$$J_{ab} = i\eta^\dagger \Gamma_{ab} \eta, \quad \Omega_{ab} = \eta^\dagger \Gamma_{ab} C\bar{\eta}. \quad (4.4)$$

The remaining possibility is not independent, since

$$i(C\bar{\eta})^\dagger \Gamma_{ab} C\bar{\eta} = i\eta^T C^{-1} \Gamma_{ab} C\bar{\eta} = i\eta^\dagger C (\Gamma_{ab})^T C^{-1} \eta = -J_{ab}. \quad (4.5)$$

Using (4.2), we can show that these forms are self-dual $*J = J$ and $*\Omega = \Omega$. By expanding the combinations $\eta \eta^\dagger$ in terms of the even elements of the Clifford algebra (i.e. in terms of products of even numbers of gamma matrices), we can refine (4.3), obtaining Fierz identities

$$\eta \eta^\dagger = \frac{1}{2} \frac{1 - \Gamma}{2} + \frac{i}{8} J^{ab} \Gamma_{ab}, \quad C\bar{\eta} (C\bar{\eta})^\dagger = \frac{1}{2} \frac{1 - \Gamma}{2} - \frac{i}{8} J^{ab} \Gamma_{ab}, \quad (4.6)$$

and

$$C\bar{\eta} \eta^\dagger = -\frac{1}{8} \Omega^{ab} \Gamma_{ab}, \quad \eta (C\bar{\eta})^\dagger = \frac{1}{8} \bar{\Omega}^{ab} \Gamma_{ab}. \quad (4.7)$$

⁵We can actually choose either sign in this relation, $\Gamma_a^T = \pm C \Gamma_a C^{-1}$; there exist conjugation matrices for both.

⁶This choice ensures that J and Ω are self-dual, which is standard in the literature, rather than anti-self-dual.

From these we can derive contraction identities, which naturally will involve the metric

$$J^{ac}J_{bc} = \delta_b^a, \quad J^{ac}\Omega_{bc} = -i\Omega_b^a, \quad \Omega^{ac}\Omega_{bc} = 0, \quad \Omega^{ac}\bar{\Omega}_{bc} = 2\delta_b^a - 2iJ_b^a, \quad (4.8)$$

and their complex conjugates. From this further contractions can be derived

$$J^{ab}J_{ab} = 4, \quad J^{ab}\Omega_{ab} = 0, \quad \Omega^{ab}\Omega_{ab} = 0, \quad \Omega^{ab}\bar{\Omega}_{ab} = 8. \quad (4.9)$$

We can also show that there are metric independent constraints relating the various two forms

$$J \wedge \Omega = \Omega \wedge \Omega = 0, \quad J \wedge J = \frac{1}{2}\Omega \wedge \bar{\Omega} \neq 0. \quad (4.10)$$

Equivalently, we can reformulate everything with the replacement of the complex two-form Ω by two real two-forms,

$$\Omega_{1ab} = \frac{1}{2}(\Omega_{ab} + \bar{\Omega}_{ab}), \quad \Omega_{2ab} = -\frac{i}{2}(\Omega_{ab} - \bar{\Omega}_{ab}). \quad (4.11)$$

The contraction identities then become

$$\begin{aligned} J^{ac}\Omega_{1bc} &= \Omega_2^a{}_b, & J^{ac}\Omega_{2bc} &= -\Omega_1^a{}_b, \\ \Omega_1^{ac}\Omega_{1bc} &= \Omega_2^{ac}\Omega_{2bc} = \delta_b^a, & \Omega_1^{ac}\Omega_{2bc} &= J_b^a, \end{aligned} \quad (4.12)$$

while the metric independent constraints read

$$J \wedge \Omega_1 = J \wedge \Omega_2 = \Omega_1 \wedge \Omega_2 = 0, \quad J \wedge J = \Omega_1 \wedge \Omega_1 = \Omega_2 \wedge \Omega_2. \quad (4.13)$$

The triplet of real two-forms $\{J, \Omega_1, \Omega_2\}$ plus the constraints (4.13) are equivalent data to the original $\{g_{ab}, \eta\}$. As a check, we have the counting

$$(\# \text{ of components of } \{g, \eta\}) = 10 + 3 = 13, \quad (4.14)$$

$$\begin{aligned} (\# \text{ of components of } \{J, \Omega_1, \Omega_2\}) - (\# \text{ of constraints}) &= 3 \times 6 - 5 \\ &= 13, \end{aligned} \quad (4.15)$$

and

$$\dim(\text{GL}(4)) - \dim(\text{SU}(2)) = 16 - 3 = 13. \quad (4.16)$$

The forms J, Ω_1, Ω_2 satisfying the constraints (4.13) build an $SU(2)$ -structure and the four-dimensional manifold is called an $SU(2)$ -structure manifold. These manifolds include, in particular, $SU(2)$ -holonomy manifolds, for which all three two forms are closed. Such manifolds are also called hyperKähler. Having an $SU(2)$ -structure is equivalent to having a metric plus a spinor of constant norm. This equivalence is again simple in one direction. Namely, having a metric and a normalized spinor one can define two forms by (4.4), that satisfy the constraints (4.13). In turn, having an $SU(2)$ -structure defines a metric and a normalized spinor. In the next subsection we illustrate how to compute the metric in terms of J, Ω_1 , and Ω_2 .

4.2 Metric from SU(2)-structure

Suppose we are given a triplet of real two-forms J , Ω_1 , and Ω_2 , which satisfy (4.13). Using the antisymmetric symbol (not tensor) $\hat{\epsilon}$, we can define

$$s_{ab} = \frac{1}{2} \hat{\epsilon}^{cdef} \Omega_{1(a|c|} \Omega_{2b)d} J_{ef}. \quad (4.17)$$

Let $s = \det(s)$, and as long as $s \neq 0$, define the metric in terms of s_{ab}

$$g_{ab} = s^{-1/6} s_{ab}. \quad (4.18)$$

With this metric and its inverse, one can verify all the contraction identities of section 4.1.

4.3 Decomposition of forms

In this section we present some useful statements about the decomposition of forms on $SU(2)$ -structure manifolds that are needed in the following subsection. Under $SO(4) \rightarrow SU(2)$, the spaces Λ^p of differential p -forms (at some given point on the manifold) decompose as

$$\Lambda^0 \cong \Lambda_1^0, \quad (4.19)$$

$$\Lambda^1 \cong \Lambda_{2\oplus 2}^1, \quad (4.20)$$

$$\Lambda^2 \cong \Lambda_{1\oplus 1\oplus 1}^2 \oplus \Lambda_3^2, \quad (4.21)$$

$$\Lambda^3 \cong \Lambda_{2\oplus 2}^3, \quad (4.22)$$

$$\Lambda^4 \cong \Lambda_0^4, \quad (4.23)$$

where the subscripts on the right-hand-side indicate the $SU(2)$ representations. For the one-forms (and also the three-forms), the individual representations **2** are not real representations (they are pseudo real), which is why we combine them in one subscript. They can be thought of as the $\pm i$ eigenspaces of the matrix $J_a{}^b$, which squares to -1 . For the two-forms, the three singlet representations correspond simply to the real forms J , Ω_1 , and Ω_2 . Note that all of these singlet two-forms are self-dual, as we saw in section 4.1. The total space of two-forms should split into anti-self-dual and self-dual subspaces,

$$\Lambda^2 \cong \Lambda_{ASD}^2 \oplus \Lambda_{SD}^2, \quad (4.24)$$

and the two spaces should have the same dimension. Remember we are talking about spaces of forms at a point, not about cohomologies, where the dimensions of these spaces are typically not equal). The fact that $\Lambda_{1\oplus 1\oplus 1}^2 \subset \Lambda_{SD}^2$ implies

$$\Lambda_{SD}^2 \cong \Lambda_{1\oplus 1\oplus 1}^2, \quad \Lambda_{ASD}^2 \cong \Lambda_3^2. \quad (4.25)$$

Using this information, let's take a brief detour to prove a useful statement that is needed in the following. Suppose we are on a compact oriented

$4k$ dimensional Riemannian manifold. For a $(2k - 1)$ -form ξ , the Hodge decomposition is

$$\xi = d\alpha + d^\dagger \beta + \gamma, \quad (4.26)$$

where α is a $(2k - 2)$ -form, β is a $(2k)$ -form, and γ is a harmonic $(2k - 1)$ -form. We claim that we can in fact take β to be self-dual. Indeed, if β is not self-dual, then take its Hodge decomposition,

$$\beta = d\mu + d^\dagger \nu + \omega, \quad (4.27)$$

and let

$$\beta' = d\mu + *d\mu = d\mu + d^\dagger (*\mu). \quad (4.28)$$

Then β' is manifestly self-dual and we also have $d^\dagger \beta' = d^\dagger \beta$, so we can write

$$\xi = d\alpha + d^\dagger \beta' + \gamma. \quad (4.29)$$

But returning now to our particular case, any self-dual two-form can be expanded in terms of J , Ω_1 and Ω_2

$$\beta' = xJ + y\Omega_1 + z\Omega_2. \quad (4.30)$$

Thus, on our $SU(2)$ -structure manifold, any one-form can be written as

$$\xi = d\lambda + d^\dagger (xJ + y\Omega_1 + z\Omega_2) + \gamma, \quad (4.31)$$

where λ , x , y , and z are functions. In components,

$$\xi_a = \nabla_a \lambda + \nabla^b (xJ_{ab} + y\Omega_{1ab} + z\Omega_{2ab}) + \gamma_a. \quad (4.32)$$

4.4 Gravitino SUSY variation and background geometry

Additional information about which $SU(2)$ -structure manifolds are relevant for M-theory/string theory follows from the gravitino supersymmetry transformation, which involves our basis of spinors

$$\nabla_a \eta = iA_a \eta + B_a C \bar{\eta}. \quad (4.33)$$

We will not need concrete expressions for the source terms A_a and B_a , which depend on the concrete theory one wishes to consider. The above is the most general expression allowed by supersymmetry. For example, if perturbative corrections to the internal geometry are allowed, these can be calculated from scattering amplitudes, and would contribute to A_a and/or B_a . These may involve Riemann tensors, covariant derivatives, and perhaps also the forms J and Ω_i . Fluxes and non-perturbative effects may also be allowed in certain theories. If additional fields are being considered, their SUSY constraints need to be checked though. Note, however, that A_a is real in order to preserve the normalization of η , but B_a may be complex,

$B_a = B_{1a} + iB_{2a}$. The twelve real components of the one-forms A_a and B_a match onto the components of the three unconstrained real three-forms dJ , $d\Omega_1$, and $d\Omega_2$. Indeed, we have

$$dJ = -2B_1 \wedge \Omega_2 - 2B_2 \wedge \Omega_1, \quad d\Omega_1 = 2A \wedge \Omega_2 + 2B_2 \wedge J, \quad d\Omega_2 = -2A \wedge \Omega_1 + 2B_1 \wedge J. \quad (4.34)$$

A_a and B_a vanish to leading order, so to leading order $dJ = d\Omega = 0$. The metric g_{ab} is thus Calabi-Yau (Kähler and Ricci flat). In fact, these manifolds are actually hyperKähler. We know that if the space is compact it is either T^4 or K3. In the case of T^4 , there are no higher curvature invariants; A_a and B_a remain zero to all orders and the solution is uncorrected. The arguments follow along the same lines as in the lower dimensional cases.

Suppose instead, that the leading order space is K3, where corrections might be allowed. See e.g [10] for a discussion of perturbative corrections in the context of $(0,4)$ sigma models or [20] for an example involving fluxes.

In the following we use primes to denote corrected quantities, and unprimed objects will correspond to the underlying K3. The integrability condition implies that the right-hand sides of (4.34) are closed, i.e. that

$$0 = -dB_1 \wedge \Omega_2 - dB_2 \wedge \Omega_1 = dA \wedge \Omega_2 + dB_2 \wedge J = -dA \wedge \Omega_1 + dB_1 \wedge J. \quad (4.35)$$

Note that this allows for the internal geometry to be non-Kähler, i.e. dJ to be non-vanishing. Let's understand the integrability constraints better. By contracting with the volume form and using the self-duality of the forms J , Ω_1 , and Ω_2 , we can recast these equations as

$$0 = \Omega_2^{ab} \nabla_a B_{1b} + \Omega_1^{ab} \nabla_a B_{2b} = \Omega_2^{ab} \nabla_a A_b + J^{ab} \nabla_a B_{2b} = \Omega_1^{ab} \nabla_a A_b - J^{ab} \nabla_a B_{1b}. \quad (4.36)$$

We now use (4.32) and the fact that the leading order J , Ω_1 , and Ω_2 are all covariantly constant and that there are no harmonic one-forms on K3, to write

$$A_a = \nabla_a \lambda_A + J_a^b \nabla_b x_A + \Omega_{1a}^b \nabla_b y_A + \Omega_{2a}^b \nabla_b z_A, \quad (4.37)$$

$$B_{1a} = \nabla_a \lambda_{B_1} + J_a^b \nabla_b x_{B_1} + \Omega_{1a}^b \nabla_b y_{B_1} + \Omega_{2a}^b \nabla_b z_{B_1}, \quad (4.38)$$

$$B_{2a} = \nabla_a \lambda_{B_2} + J_a^b \nabla_b x_{B_2} + \Omega_{1a}^b \nabla_b y_{B_2} + \Omega_{2a}^b \nabla_b z_{B_2}. \quad (4.39)$$

In terms of these components, the integrability conditions become simply

$$0 = \nabla^2 (z_{B_1} + y_{B_2}) = \nabla^2 (z_A + x_{B_2}) = \nabla^2 (y_A - x_{B_1}). \quad (4.40)$$

Note that we are free to shift any of the λ 's, x 's, y 's, or z 's by constants, since this will not change the one-forms. Hence, since constants are also the only harmonic functions on a compact connected space, the fully general solution to the integrability conditions is

$$0 = z_{B_1} + y_{B_2} = z_A + x_{B_2} = y_A - x_{B_1}. \quad (4.41)$$

Now put these decompositions into the equation (4.34)

$$\begin{aligned}
(dJ')_{abc} &= -6B_1[a\Omega_{2bc}] - 6B_2[a\Omega_{1bc}] \\
&= -6\left(\nabla_{[a}\lambda_{B_1} + J_{[a}{}^d\nabla_{|d|}x_{B_1} + \Omega_{1[a}{}^d\nabla_{|d|}y_{B_1} + \Omega_{2[a}{}^d\nabla_{|d|}z_{B_1}\right)\Omega_{2bc]} \\
&\quad -6\left(\nabla_{[a}\lambda_{B_2} + J_{[a}{}^d\nabla_{|d|}x_{B_2} + \Omega_{1[a}{}^d\nabla_{|d|}y_{B_2} + \Omega_{2[a}{}^d\nabla_{|d|}z_{B_2}\right)\Omega_{1bc]}.
\end{aligned} \tag{4.42}$$

To proceed further, note that we have identities such as

$$J_{[a}{}^d\nabla_{|d|}x\Omega_{2bc]} = -\nabla_{[a}x\Omega_{1bc]}, \tag{4.43}$$

and others related by permutation of the three-forms, which can be derived using the self-duality of the forms and the contraction identities of section 4.1. From this one can show that the equation above becomes in fact

$$dJ' = 2d[(-y_{B_1} + z_{B_2})J + (x_{B_1} - \lambda_{B_2})\Omega_1 + (-\lambda_{B_1} - x_{B_2})\Omega_2]. \tag{4.44}$$

Note that to reach this form we must make use of the fact $z_{B_1} + y_{B_2} = 0$ which we derived from integrability.

Thus, we should make an ansatz for the $SU(2)$ structure

$$J' = J + 2(-y_{B_1} + z_{B_2})J + 2(x_{B_1} - \lambda_{B_2})\Omega_1 + 2(-\lambda_{B_1} - x_{B_2})\Omega_2 + da, \tag{4.45}$$

where a is some arbitrary one-form.

Similarly with the other forms, we find that we should write

$$\Omega'_1 = \Omega_1 + 2(y_A + \lambda_{B_2})J + 2(-x_A + z_{B_2})\Omega_1 + 2(\lambda_A - y_{B_2})\Omega_2 + db, \tag{4.46}$$

$$\Omega'_2 = \Omega_2 + 2(z_A + \lambda_{B_1})J + 2(-\lambda_A + z_{B_1})\Omega_1 + 2(-x_A - y_{B_1})\Omega_2 + dc, \tag{4.47}$$

where again b and c are one-forms. These expressions for J' , Ω'_1 , and Ω'_2 now automatically solve the variation equations (4.34), but we still have to do some work to ensure that the constraints (4.13). These will constrain the one-forms a , b , and c .

To tackle this next step, let us use (4.32) again to expand a , b , and c ,

$$a = \nabla_a\lambda_1 + J_a{}^b\nabla_b x_1 + \Omega_{1a}{}^b\nabla_b y_1 + \Omega_{2a}{}^b\nabla_b z_1, \tag{4.48}$$

$$b = \nabla_a\lambda_2 + J_a{}^b\nabla_b x_2 + \Omega_{1a}{}^b\nabla_b y_2 + \Omega_{2a}{}^b\nabla_b z_2, \tag{4.49}$$

$$c = \nabla_a\lambda_3 + J_a{}^b\nabla_b x_3 + \Omega_{1a}{}^b\nabla_b y_3 + \Omega_{2a}{}^b\nabla_b z_3. \tag{4.50}$$

In terms of these, the orthogonality constraints become

$$0 = \nabla^2(y_1 + x_2) - 4(y_A + x_{B_1}), \tag{4.51}$$

$$0 = \nabla^2(z_1 + x_3) - 4(z_A - x_{B_2}), \tag{4.52}$$

$$0 = \nabla^2(z_2 + y_3) - 4(z_{B_1} - y_{B_2}), \tag{4.53}$$

while the normalization conditions lead to

$$\nabla^2 x_1 - 4(-y_{B_1} + z_{B_2}) = \nabla^2 y_2 - 4(-x_A + z_{B_2}) = \nabla^2 z_3 - 4(-x_A - y_{B_1}). \quad (4.54)$$

These give five Poisson equations for the nine functions x_i , y_i , and z_i , where i runs from one to three. Note that the λ_i of course drop out, since they don't appear in da , db , or dc . Furthermore, these Poisson equations can always be solved, once we make use of the freedom to shift the source terms by constants.

To summarize, we can pick four of the functions, say y_1 , z_1 , z_2 , and y_2 , arbitrarily, and then fix the remaining functions by solving Poisson equations,

$$\nabla^2 x_1 = \nabla^2 y_2 + 4(x_A - y_{B_1}), \quad (4.55)$$

$$\nabla^2 x_2 = -\nabla^2 y_1 + 4(y_A + x_{B_1}), \quad (4.56)$$

$$\nabla^2 x_3 = -\nabla^2 z_1 + 4(z_A - x_{B_2}), \quad (4.57)$$

$$\nabla^2 y_3 = -\nabla^2 z_2 + 4(z_{B_1} - y_{B_2}), \quad (4.58)$$

$$\nabla^2 z_3 = \nabla^2 y_2 + 4(-y_{B_1} - z_{B_2}). \quad (4.59)$$

Here all quantities labeled with subindices A , B need external input and act as source terms, coming from string theory. Their explicit form is not relevant for our discussion. Once this is done, then the corrected SU(2)-structure, or equivalently the corrected metric and spinor, solve the full system of equations. Since we can always do this, it means that we can always correct the K3 metric in order to preserve SUSY to this order. The new physical metric is no longer Ricci-flat nor Kähler (for general A_a and B_a).

Let's understand this solution in a bit more detail. First of all, what is the significance of the four arbitrary functions? To shed light on their role, let us compute the correction to the metric. By taking variations of the contraction identity

$$g^{cd}\Omega_{1ac}\Omega_{2bd} = J_{ab}, \quad (4.60)$$

we derive

$$-\Omega_{1a}{}^c\Omega_{2b}{}^d\delta g_{cd} + \Omega_{2b}{}^c\delta\Omega_{1ac} + \Omega_{1a}{}^c\delta\Omega_{2bc} = \delta J_{ab}, \quad (4.61)$$

which we can rearrange to get

$$\delta g_{ab} = \Omega_{1a}{}^c\delta\Omega_{1bc} + \Omega_{2b}{}^c\delta\Omega_{2ac} - \Omega_{1a}{}^c\Omega_{2b}{}^d\delta J_{cd}. \quad (4.62)$$

Plugging in J' , Ω'_1 , and Ω'_2 ,

$$\begin{aligned}\delta g_{ab} = & -4x_A g_{ab} + \left(\Omega_{1(a}{}^c \Omega_{1b)}{}^d + \Omega_{2(a}{}^c \Omega_{2b)}{}^d \right) \nabla_c \nabla_d (x_1 - y_2) \\ & + \left(\delta_{(a}^c \Omega_{2b)}{}^d - J_{(a}{}^c \Omega_{1b)}{}^d \right) \nabla_c \nabla_d (x_2 + y_1) \\ & + \left(-\delta_{(a}^c \Omega_{1b)}{}^d - J_{(a}{}^c \Omega_{2b)}{}^d \right) \nabla_c \nabla_d (x_3 + z_1) \\ & + \left(\delta_{(a}^c J_{b)}{}^d - \Omega_{1(a}{}^c \Omega_{2b)}{}^d \right) \nabla_c \nabla_d (y_3 + z_2) \\ & + \left(-\delta_{(a}^c \delta_{b)}^d - \Omega_{2(a}{}^c \Omega_{2b)}{}^d \right) \nabla_c \nabla_d (z_3 - y_2) + \nabla_a \xi_b + \nabla_b \xi_a, \quad (4.63)\end{aligned}$$

where we have defined

$$\xi_a = -\nabla_a y_2 - J_a{}^b \nabla_b z_2 + \Omega_{1a}{}^b \nabla_b z_1 - \Omega_{2a}{}^b \nabla_b y_1. \quad (4.64)$$

Comparing to (4.32), we see that by choosing y_1 , z_1 , y_2 , and z_2 we can get the most general possible vector ξ_a , and that any choice of ξ_a generates an infinitesimal diffeomorphism of the metric. So the arbitrariness of the choices of these four functions simply corresponds to the possibility of arbitrary infinitesimal changes of coordinates.

Now let us briefly restrict to the situation when $B_a = 0$. In this case all the functions with B_1 or B_2 subscripts vanish, and the integrability conditions imply that $y_A = z_A = 0$, leaving only λ_A and x_A . Taking $y_1 = z_1 = y_2 = z_2 = 0$, the Poisson equations are solved by setting $x_2 = x_3 = y_3 = z_3 = 0$, and solving

$$\nabla^2 x_1 = 4x_A. \quad (4.65)$$

The corrected SU(2)-structure is in this simpler case

$$J' = J + da, \quad \Omega'_1 = \Omega_1 + 2\lambda_A \Omega_2 - 2x_A \Omega_1, \quad \Omega'_2 = \Omega_2 - 2\lambda_A \Omega_1 - 2x_A \Omega_2, \quad (4.66)$$

where

$$a_a = J_a{}^b \nabla_b x_1. \quad (4.67)$$

The corrected metric remains Kähler, $dJ' = 0$, but Ω'_1 and Ω'_2 are no longer necessarily closed. As mentioned before, the corrected physical metric is no longer Ricci-flat. Therefore the new manifold is not Calabi-Yau in the metric sense. It is nevertheless, Calabi-Yau in the topological sense, as it is Kähler and has $c_1(M) = 0$.

The role of x_1 is somewhat clarified if we adopt local complex coordinates in which $J_i{}^{\bar{j}} = i\delta_i^{\bar{j}}$, $J_{\bar{i}}{}^j = -i\delta_{\bar{i}}^j$, and $J_i{}^{\bar{j}} = J_{\bar{i}}{}^j = 0$. Then if we define a complex scalar $\varphi = \lambda_A + ix_A$, we have $A_i = \partial_i \varphi$, $A_{\bar{i}} = \partial_{\bar{i}} \bar{\varphi}$. The complex two-form becomes $\Omega' = (1 - 2i\bar{\varphi})\Omega$. Finally, we have $a_i = i\partial_i x_1$, $a_{\bar{i}} = -i\partial_{\bar{i}} x_1$, and thus

$$J'_{ij} = J'_{\bar{i}\bar{j}} = 0, \quad J'_{i\bar{j}} = J_{i\bar{j}} - 2i\partial_i \partial_{\bar{j}} x_1. \quad (4.68)$$

In other words, x_1 is simply a shift in the Kähler potential.

Finally, returning to the general case, notice that λ_A , λ_{B_1} , and λ_{B_2} appeared in a rather trivial way, dropping out of the Poisson equations. In fact, we could have removed the λ 's by taking a different choice of nowhere vanishing positive chirality spinor. Indeed, we can a priori make an $SU(2)$ rotation in the space spanned by η and $C\bar{\eta}$, to define a new spinor

$$\hat{\eta} = \alpha\eta + \beta C\bar{\eta}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (4.69)$$

If η obeyed the SUSY variation (4.33), then $\hat{\eta}$ will obey a similar equation, but where A_a and B_a get modified. Choosing appropriate infinitesimal rotations, we can in fact shift A_a and B_a by exact quantities, thus effectively shifting the λ 's.

5 Six dimensions

M-theory/string theory corrections, in particular perturbative effects coming from higher derivatives or α' -corrections, are possible upon compactification on a 6D Kähler manifold [6], [8] and [7]. In this case the physical metric may no longer be Ricci flat, but nevertheless $c_1(M) = 0$. The corrected geometry has $SU(3)$ -structure, rather than $SU(3)$ -holonomy, as we show explicitly in the following. See [16] and references therein for related discussions.

5.1 Spinors and bispinors

In six Euclidean dimensions we can pick gamma-matrices Γ_a which are imaginary and antisymmetric. We can define the chirality operator

$$\Gamma = \frac{i}{6!} \epsilon^{abcdef} \Gamma_{abcdef}, \quad (5.1)$$

which is also imaginary antisymmetric. Here ϵ^{abcdef} is the antisymmetric tensor with entries $\pm 1/\sqrt{g}$.

Suppose we have a nowhere vanishing real spinor η . This reduces the structure group from $SO(6)$ to $SU(3)$ which leaves the spinor invariant. We are free to normalize the spinor so that $\eta^T \eta = 1$ pointwise. Then, again pointwise, one choice of basis for the eight dimensional space of real spinors is $\{\eta, i\Gamma\eta, i\Gamma_a\eta\}$. The corresponding dual basis is given by $\{\eta^T, -i\eta^T\Gamma, -i\eta^T\Gamma^a\}$. The completeness relation for this basis is the Fierz identity

$$1 = \eta\eta^T + \Gamma\eta\eta^T\Gamma + \Gamma^a\eta\eta^T\Gamma_a. \quad (5.2)$$

Along with the zero form 1 and the six-form ϵ_{abcdef} , the existence of η allows us to define the following real forms, which by construction are invariant under the $SU(3)$ -structure group

$$J_{ab} = -i\eta^T\Gamma_{ab}\Gamma\eta, \quad L_{abcd} = -\eta^T\Gamma_{abcd}\eta, \quad (5.3)$$

$$\Omega_{1abc} = -i\eta^T \Gamma_{abc} \eta, \quad \Omega_{2abc} = -\eta^T \Gamma_{abc} \Gamma \eta. \quad (5.4)$$

It is easy to verify that these forms satisfy the duality relations

$$*J = L, \quad *L = J, \quad *\Omega_1 = -\Omega_2, \quad *\Omega_2 = \Omega_1. \quad (5.5)$$

Note that $*$ squares to one on even forms but to minus one on odd forms. Using (5.2) and manipulations with epsilon tensors, we can show that the four-form is not independent

$$L_{abcd} = 3J_{[ab}J_{cd]}. \quad (5.6)$$

Because of this, we will have no need to refer to L_{abcd} explicitly in the rest of the note.

Using the Fierz identity (5.2) we can derive contraction identities,

$$J^{ac}J_{bc} = \delta_b^a, \quad J^{ad}\Omega_{1bcd} = -\Omega_2^a{}_{bc}, \quad J^{ad}\Omega_{2bcd} = \Omega_1^a{}_{bc}, \quad (5.7)$$

$$\Omega_1^{abe}\Omega_{1cde} = 2\delta_{[c}^{[a}\delta_{d]}^{b]} - 2J_{[c}^{[a}J_{d]}^{b]} = \Omega_2^{abe}\Omega_{2cde}, \quad \Omega_1^{abe}\Omega_{2cde} = 4\delta_{[c}^{[a}J_{d]}^{b]}. \quad (5.8)$$

$$J^{ab}J_{ab} = 6, \quad J^{bc}\Omega_{1abc} = J^{bc}\Omega_{2abc} = 0, \quad \Omega_1^{acd}\Omega_{1bcd} = \Omega_2^{acd}\Omega_{2bcd} = 4\delta_b^a, \quad (5.9)$$

$$\Omega_1^{acd}\Omega_{2bcd} = 4J_b^a, \quad \Omega_1^{abc}\Omega_{1abc} = \Omega_2^{abc}\Omega_{2abc} = 24, \quad \Omega_1^{abc}\Omega_{2abc} = 0. \quad (5.10)$$

Similarly we can obtain the orthogonality relations and normalization constraints

$$J_{[ab}\Omega_{1cde]} = J_{[ab}\Omega_{2cde]} = 0, \quad \Omega_{1[abc}\Omega_{2def]} = 3J_{[ab}J_{cd}J_{ef]}. \quad (5.11)$$

In terms of maps between variables, a choice of forms $\{J, \Omega_1\}$, satisfying (5.11) is equivalent to the metric and normalized spinor, $\{g, \eta\}$. Here Ω_2 is constructed from J and Ω_1 , as we discuss in the next subsection. As a check, we have the counting

$$(\# \text{ of components of } \{g, \eta\}) = 21 + 7 = 28, \quad (5.12)$$

$$\begin{aligned} & (\# \text{ of components of } \{J, \Omega_1, \Omega_2\}) - (\# \text{ of constraints}) \\ &= (15 + 20) - (6 + 1) = 28, \end{aligned} \quad (5.13)$$

and

$$\dim(\text{GL}(6)) - \dim(\text{SU}(3)) = 36 - 8 = 28. \quad (5.14)$$

5.2 Metric from $SU(3)$ -structure

Suppose we are given a real two-form J and a real three-form $\tilde{\Omega}_1$ satisfying $J \wedge \tilde{\Omega}_1 = 0$. The tilde is because Ω_1 is related to $\tilde{\Omega}_1$ by the rescaling below. Using the antisymmetric symbol (not tensor) $\hat{\epsilon}$, define a symmetric matrix in terms of the $SU(3)$ -structure,

$$s_{ab} = \hat{\epsilon}^{cdefgh} \tilde{\Omega}_{1acd} \tilde{\Omega}_{1bef} J_{gh}. \quad (5.15)$$

Let $s = \det(s_{ab})$ and as long as $s \neq 0$ define

$$\tilde{g}_{ab} = \frac{1}{2} s^{-\frac{1}{8}} s_{ab}. \quad (5.16)$$

Now define a scaling factor

$$\lambda = \frac{1}{6} \tilde{g}^{ab} \tilde{g}^{cd} J_{ac} J_{bd}, \quad (5.17)$$

and finally we obtain the three form and metric,

$$\Omega_{1abc} = \lambda \tilde{\Omega}_{1abc}, \quad g_{ab} = \lambda^{1/2} \tilde{g}_{ab}. \quad (5.18)$$

From (5.2) we see that Ω_{2abc} is not independent

$$\Omega_{2abc} = -J_{ad} g^{de} \Omega_{1bce}. \quad (5.19)$$

With these expressions we can verify that the various contractions (5.7)-(5.10) are satisfied.

It will be useful to have the relations above to linear order in perturbations around a background solution given by $\{J, \Omega_1, \Omega_2, g\}$. Suppose we are given perturbations δJ_{ab} and $\delta \Omega_{1abc}$. To ensure that the constraints are satisfied, these must obey (here indices are raised and lowered with the uncorrected metric)

$$J^{bc} \delta \Omega_{1abc} + \Omega_{1a}{}^{bc} \delta J_{bc} = 0, \quad 6J^{ab} \delta J_{ab} - \Omega_1^{abc} \delta \Omega_{1abc} = 0. \quad (5.20)$$

The first condition guarantees $J \wedge \Omega_1 = 0$ and the second $J \wedge J \wedge J = \frac{3}{2} \Omega_1 \wedge \Omega_2$, even once the perturbations are included. As long as these relations are obeyed by δJ_{ab} and $\delta \Omega_{1abc}$, then we have

$$\delta g_{ab} = -\frac{1}{2} g_{ab} J^{cd} \delta J_{cd} + J_{(a}{}^c \delta J_{b)c} + \frac{1}{2} \Omega_{1(a}{}^{cd} \delta \Omega_{1b)cd}, \quad (5.21)$$

$$\delta \Omega_{2abc} = -J_a{}^d \delta \Omega_{1bcd} - \Omega_{1bc}{}^d \delta J_{ad} + J_a{}^d \Omega_{1bc}{}^e \delta g_{de}. \quad (5.22)$$

5.3 Decomposition of forms

Similarly as we did for $n = 4$, we present the decomposition of forms into $SU(3)$ representations and derive some useful formulas needed to evaluate the gravitino SUSY transformation. Under $SO(6) \rightarrow SU(3)$, the spaces of differential forms decompose as

$$\Lambda^0 = \Lambda_1^0, \quad (5.23)$$

$$\Lambda^1 = \Lambda_{3 \oplus \bar{3}}^1, \quad (5.24)$$

$$\Lambda^2 = \Lambda_1^2 \oplus \Lambda_{3 \oplus \bar{3}}^2 \oplus \Lambda_8^2, \quad (5.25)$$

$$\Lambda^3 = \Lambda_{1 \oplus 1}^3 \oplus \Lambda_{3 \oplus \bar{3}}^3 \oplus \Lambda_{6 \oplus \bar{6}}^3, \quad (5.26)$$

$$\Lambda^4 = \Lambda_1^4 \oplus \Lambda_{3 \oplus \bar{3}}^4 \oplus \Lambda_8^4, \quad (5.27)$$

$$\Lambda^5 = \Lambda_{3 \oplus \bar{3}}^5, \quad (5.28)$$

$$\Lambda^6 = \Lambda_1^6. \quad (5.29)$$

At any given point the singlet spaces Λ_1^0 , Λ_1^2 , $\Lambda_{1 \oplus 1}^3$, Λ_1^4 , and Λ_1^6 are spanned by 1, J , Ω_1 and Ω_2 , $J \wedge J$, and $J \wedge J \wedge J$, respectively.

We can derive projectors for the two-, three-, and four-forms. For a two-form α_{ab} , we have

$$\pi_1(\alpha)_{ab} = \frac{1}{6} J_{ab} J^{cd} \alpha_{cd}, \quad (5.30)$$

$$\pi_{3 \oplus \bar{3}}(\alpha)_{ab} = \frac{1}{2} \alpha_{ab} - \frac{1}{2} J_{[a}^c J_{b]}^d \alpha_{cd}, \quad (5.31)$$

$$\pi_8(\alpha)_{ab} = \frac{1}{2} \alpha_{ab} + \frac{1}{2} J_{[a}^c J_{b]}^d \alpha_{cd} - \frac{1}{6} J_{ab} J^{cd} \alpha_{cd}, \quad (5.32)$$

while the remaining projectors are written for completeness in the appendix. The former projector is all we need in the following.

Next we would like to write an arbitrary one-form on a Calabi-Yau three-fold, along the lines of (4.32). On a Calabi-Yau three-fold there are no harmonic one-forms, so the Hodge decomposition for one-forms is

$$\xi = d\lambda + d^\dagger \beta. \quad (5.33)$$

To see that we can make a convenient choice for β note that we have the following equation for one-forms α ,

$$d^\dagger (\pi_8 - \pi_{3 \oplus \bar{3}} - 2\pi_1) d\alpha = 0, \quad (5.34)$$

which follows by considering the explicit form of the projection operators. Then we can do a Hodge decomposition of β ,

$$\beta = d\alpha + d^\dagger \omega + \gamma, \quad (5.35)$$

where γ is harmonic. Let

$$\beta' = (3\pi_1 + 2\pi_{3\oplus\bar{3}}) d\alpha = \beta - (\pi_8 - \pi_{3\oplus\bar{3}} - 2\pi_1) d\alpha - d^\dagger \omega - \gamma. \quad (5.36)$$

Then clearly we have $\pi_8 \beta' = 0$ and $d^\dagger \beta' = d^\dagger \beta$. Thus we can always assume that there is no 8 piece in β . The decompositions then take the explicit form

$$\beta_{ab} = xJ_{ab} + \Omega_{1abc}v^c, \quad (5.37)$$

and

$$\xi_a = \nabla_a \lambda + J_a{}^b \nabla_b x + \Omega_{1a}{}^{bc} \nabla_b v_c. \quad (5.38)$$

We can actually go further. Since only dv appears above, we can always shift v_a by something exact to arrange that v is co-closed, $\nabla^a v_a = 0$. Similarly, we can show that it is always possible to shift v to arrange that $J \cdot v$ is also co-closed, i.e. that $J^{ab} \nabla_a v_b = 0$.

5.4 Gravitino SUSY variation and background geometry

The most general form of the gravitino SUSY transformation expresses η in terms of the basis of spinors

$$\nabla_a \eta = iA_a \Gamma \eta + iB_{ab} \Gamma^b \eta. \quad (5.39)$$

Here A_a and B_{ab} may involve a derivative expansion in terms of Riemann tensors, non-perturbative effects or/and fluxes. E.g. it is known that the α'^3 perturbative contribution is non-vanishing for compactifications on Ricci flat Kähler manifolds (see [21], [8], [6] and [7]). From the previous equation we can compute the derivatives for the $SU(3)$ -structure

$$\nabla_a J_{bc} = -2i\eta^T \Gamma_{bc} \Gamma \nabla_a \eta = 2B_a{}^d \Omega_{2bcd}, \quad (5.40)$$

$$\nabla_a \Omega_{1bcd} = -2i\eta^T \Gamma_{bcd} \Gamma \nabla_a \eta = -2A_a \Omega_{2bcd} - 2B_a{}^e L_{bcde}, \quad (5.41)$$

$$\nabla_a \Omega_{2bcd} = -2\eta^T \Gamma_{bcd} \Gamma \nabla_a \eta = 2A_a \Omega_{1bcd} - 6B_{a[b} J_{cd]}, \quad (5.42)$$

$$(5.43)$$

Antisymmetrizing, this gives dJ , $d\Omega_1$ and $d\Omega_2$, in terms of A_a and B_{ab} . In general we can expand these forms in their $SU(3)$ representations. These are not all independent, because of the relations obeyed by the underlying forms J , Ω_1 , and Ω_2 . The independent components transform as

$$[2 \times \mathbf{1}] \oplus [2 \times (\mathbf{3} \oplus \bar{\mathbf{3}})] \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus [2 \times \mathbf{8}], \quad (5.44)$$

which is a subset of the representations in $\Lambda^3 \oplus \Lambda^4 \oplus \Lambda^4$. These are also known as the $SU(3)$ -structure torsion classes W_i , and they are in one-to-one correspondence with the forty-two components of A_a and B_{ab} . Note that α , β , and γ are closed forms by construction. This follows by using the integrability condition for the SUSY variation.

Given A_a and B_{ab} and an uncorrected solution for J , Ω_1 and Ω_2 satisfying the constraints (5.11), our goal is to find $J' = J + \delta J$ and $\Omega'_1 = \Omega_1 + \delta\Omega_1$ such that

$$\begin{aligned} dJ' &= \alpha, & d\Omega'_1 &= \beta, & d\Omega'_2 &= \gamma, \\ J' \wedge \Omega'_1 &= 0, & J' \wedge J' \wedge J' &= \frac{3}{2}\Omega'_1 \wedge \Omega'_2, \end{aligned} \quad (5.45)$$

Here Ω'_2 is derived from J' and Ω'_1 and α , β and γ are constructed from A_a and B_{ab} , as we show for the $B_{ab} = 0$ case next.

Take $B_{ab} = 0$ and interpret A_a as a one-form. In this case the corrected two form J' is closed, while the $d\Omega'$ components receive a correction,

$$\alpha = 0, \quad \beta = -2A \wedge \Omega_2, \quad \gamma = 2A \wedge \Omega_1. \quad (5.46)$$

As a result the corrected manifold remains Kähler. To work out the explicit expression for the one form A notice that the integrability conditions say

$$dA \wedge \Omega_1 = dA \wedge \Omega_2 = 0. \quad (5.47)$$

Taking this into account and using the decomposition (5.38) for A , we derive the expression

$$A_a = \nabla_a \lambda_A + J_a{}^b \nabla_b x_A. \quad (5.48)$$

Just as in four dimensions, the integrability conditions require the term with v in the expansion of A_a to vanish.

Similarly as in 4D this points to an ansatz

$$J' = J + da, \quad \Omega'_1 = \Omega_1 - 2\lambda_0 \Omega_2 + 2x_0 \Omega_1 + db. \quad (5.49)$$

In components,

$$\delta J_{ab} = 2\nabla_{[a} a_{b]}, \quad \delta\Omega_{1abc} = -2\lambda_A \Omega_{2abc} + 2x_A \Omega_{1abc} + 3\nabla_{[a} b_{bc]}. \quad (5.50)$$

The goal is now to pick a_a and b_{ab} such that the linearized constraints (5.20) are obeyed and

$$4\nabla_{[a} \delta\Omega_{2bcd]} = -8\Omega_{1[abc} \nabla_d] \lambda_A - 8\Omega_{2[abc} \nabla_d] x_A, \quad (5.51)$$

where $\delta\Omega_{2abc}$ is defined by (5.22).

First we will exhibit a particular solution to this system, by taking $b_{ab} = 0$ and $a_a = J_a{}^b \nabla_b \rho$, for some function ρ . Then it is easy to check that the first constraint of (5.20) is satisfied, while the second becomes a Poisson equation

$$\nabla^2 \rho + 4x_A = 0. \quad (5.52)$$

As is familiar from the four-dimensional case, we can always solve this condition, with x_A , possibly shifted by a constant, acting as a source term.

Now we compute

$$\delta g_{ab} = -\nabla_{(a}\nabla_{b)}\rho - J_{(a}{}^c J_{b)}{}^d \nabla_c \nabla_d \rho, \quad (5.53)$$

$$\delta \Omega_{2abc} = 2\lambda_A \Omega_{1abc} + 2x_A \Omega_{2abc}, \quad (5.54)$$

where in both cases we have used our constraint equation (5.52). It is now simple to check that (5.51) is also satisfied, so we have a solution to the full system of equations.

In fact, we can exhibit a whole six-dimensional space of solutions. Let v^a be an arbitrary vector on the manifold, and take

$$a_a = J_a{}^b \nabla_b \rho - J_{ab} v^b, \quad b_{ab} = \Omega_{1abc} v^c. \quad (5.55)$$

Then we can check that v^a drops out of the constraints (5.20), so we have only the Poisson equation (5.52) for ρ again. Then we also have

$$\delta g_{ab} = -\nabla_{(a}\nabla_{b)}\rho - J_{(a}{}^c J_{b)}{}^d \nabla_c \nabla_d \rho + \nabla_a v_b + \nabla_b v_a, \quad (5.56)$$

$$\delta \Omega_{2abc} = 2\lambda_A \Omega_{1abc} + 2x_A \Omega_{2abc} - \Omega_{2[ab}{}^d \nabla_{c]} v_d. \quad (5.57)$$

Since the term added to $\delta \Omega_2$ is exact, it of course drops out of the remaining equation (5.51), and so we still have a solution to the full system of equations. Moreover, from the form of the addition to δg_{ab} , we see that v^a acts simply as an infinitesimal diffeomorphism. We expect this to be the full space of solutions.

Acknowledgement

This work was supported by the grants PHY-1214344 and NSF Focused Research Grant DMS-1159404 and by the George P. and Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy. We thank J. Caldeira, A. Royston, and S. Sethi for useful discussions.

A Useful formulae

For completeness we have collected some additional formulas for the $n = 6$ case in this appendix.

1. To make contact with the approach using complex spinors, note that we can define a complex chiral spinor by

$$\zeta = \frac{1}{\sqrt{2}} (1 + \Gamma) \eta. \quad (A.1)$$

This satisfies

$$\Gamma \zeta = \zeta, \quad \Gamma \bar{\zeta} = -\bar{\zeta}, \quad \zeta^\dagger \zeta = 1. \quad (A.2)$$

Moreover, using the contraction identities we can show that

$$\left(i\Gamma_a - J_a{}^b\Gamma_b\right)\zeta = 0, \quad \left(i\Gamma_a + J_a{}^b\Gamma_b\right)\bar{\zeta} = 0. \quad (\text{A.3})$$

If we picked holomorphic coordinates by splitting into $+i$ and $-i$ eigenspaces of the matrix $J_a{}^b$ (which squares to minus one), then this states that

$$\Gamma_i\zeta = 0, \quad \Gamma_i\bar{\zeta} = 0. \quad (\text{A.4})$$

Conversely, starting with a nowhere-vanishing complex chiral spinor ζ , which we can normalize so that $\zeta^\dagger\zeta = 1$, we can define a real spinor

$$\eta = \sqrt{2}\text{Re}(\zeta), \quad (\text{A.5})$$

which implies (A.1). In particular η is also nowhere-vanishing.

2. The projectors for the three-form β_{abc} are,

$$\pi_{1\oplus 1}(\beta)_{abc} = \frac{1}{24} \left(\Omega_{1abc}\Omega_1^{def}\beta_{def} + \Omega_{2abc}\Omega_2^{def}\beta_{def} \right), \quad (\text{A.6})$$

$$\pi_{3\oplus \bar{3}}(\beta)_{abc} = \frac{3}{4} J_{[ab}J^{de}\beta_{c]de}, \quad (\text{A.7})$$

$$\begin{aligned} \pi_{6\oplus \bar{6}}(\beta)_{abc} = & \beta_{abc} - \frac{3}{4} J_{[ab}J^{de}\beta_{c]de} \\ & - \frac{1}{24} \left(\Omega_{1abc}\Omega_1^{def}\beta_{def} + \Omega_{2abc}\Omega_2^{def}\beta_{def} \right) \end{aligned} \quad (\text{A.8})$$

For a four-form γ_{abcd} the projectors take the form,

$$\pi_1(\gamma)_{abcd} = -\frac{1}{3}\gamma_{abcd} + J_{[ab}J^{ef}\gamma_{cd]ef}, \quad (\text{A.9})$$

$$\pi_{3\oplus \bar{3}}(\gamma)_{abcd} = \gamma_{abcd} - 3J_{[a}{}^e J_b{}^f \gamma_{cd]ef}, \quad (\text{A.10})$$

$$\pi_8(\gamma)_{abcd} = \frac{1}{3}\gamma_{abcd} - J_{[ab}J^{ef}\gamma_{cd]ef} + 3J_{[a}{}^e J_b{}^f \gamma_{cd]ef}. \quad (\text{A.11})$$

References

- [1] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B **258** (1985) 46.
- [2] E. Calabi, Algebraic geometry and topology: a symposium in honor of S. Lefschetz (Princeton University Press, 1957) 78.
- [3] S. T. Yau, Proc. Nat. Acad. Sci. **74**, 1798 (1977).
- [4] M. R. Douglas, arXiv:1503.02899 [hep-th].
- [5] E. Witten, Nucl. Phys. B **268**, 79 (1986).
- [6] D. J. Gross and E. Witten, Nucl. Phys. B **277** (1986) 1.
- [7] P. Candelas, M. D. Freeman, C. N. Pope, M. F. Sohnius and K. S. Stelle, Phys. Lett. B **177**, 341 (1986).
- [8] D. Nemeschansky and A. Sen, Phys. Lett. B **178**, 365 (1986).
- [9] P. S. Howe and G. Papadopoulos, Nucl. Phys. B **289**, 264 (1987).
- [10] P. S. Howe and G. Papadopoulos, Nucl. Phys. B **381**, 360 (1992) [hep-th/9203070].
- [11] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, Commun. Math. Phys. **247**, 421 (2004) [hep-th/0205050].
- [12] D. Martelli and J. Sparks, Phys. Rev. D **68**, 085014 (2003) [hep-th/0306225].
- [13] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, JHEP **0408**, 046 (2004) [hep-th/0406137].
- [14] A. K. Kashani-Poor and R. Minasian, JHEP **0703**, 109 (2007) [hep-th/0611106].
- [15] S. Fidanza, R. Minasian and A. Tomasiello, Commun. Math. Phys. **254**, 401 (2005) [hep-th/0311122].
- [16] K. S. Stelle, J. Phys. Conf. Ser. **33**, 182 (2006).
- [17] H. Lu, C. N. Pope and K. S. Stelle, Nucl. Phys. B **741**, 17 (2006) [hep-th/0509057].
- [18] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, JHEP **1411**, 160 (2014) [arXiv:1407.7542 [hep-th]].
- [19] K. Becker, D. Robbins and E. Witten, JHEP **1406**, 051 (2014) [arXiv:1404.2460 [hep-th]].
- [20] A. Strominger, Nucl. Phys. B **274**, 253 (1986).
- [21] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, Nucl. Phys. B **277**, 388 (1986).